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A polynomial invariant of rational homology 3-spheres

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Introduction

In [21] Witten proposed topological invariants of a compact oriented 3-manifold M, what we call quantum G invariants, which are formally expressed by

$$Z_k(M,G) = \int \exp\left(\frac{\sqrt{-1}k}{4\pi}\int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3}A^3\right)\right)\mathscr{D}A$$

where k is an integer and G is a compact Lie group and the first integral is over all G connections, what we call the Feynman path integral. Since the set of G connections is infinitely dimensional, this integral is purely formal from a mathematical viewpoint. By perturbation theory the asymptotic formula of $Z_k(M,G)$ was studied for large k limit $[1,2,12]$. It is power series in k^{-1} and each coefficient is a sum of contributions from flat connections.

In [17] Reshetikhin and Turaev gave a mathematical definition of quantum $SU(2)$ invariants for closed oriented 3-manifolds and positive k. We denote them by $\tau_r(M)$, which correspond to $Z_{r-2}(M,SU(2))/Z_{r-2}(S^3,SU(2))$. For odd r, Kirby and Melvin [9] defined quantum $SO(3)$ invariants $\tau'_r(M)$ and showed that $\tau_r(M)$ splits into the product of $\tau'_r(M)$ and $\tau_3(M)$.

Our motivation is to construct power series invariant of a closed oriented 3-manifold M corresponding to the above asymptotic formula of $Z_k(M,SO(3))$ by expanding $\tau'_r(M)$ into power series in $q-1$ where $q = \exp(2\pi\sqrt{-1}/r)$.

Throughout this paper we put r to be an odd prime. In this case H. Murakami [14] showed that $\tau'_r(M)$ belongs to $\mathbb{Z}[q]$. Hence we can put

$$\tau'_r(M) = a_{r,0} + a_{r,1}(q-1) + a_{r,2}(q-1)^2 + \cdots + a_{r,N}(q-1)^N ,$$

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with some integers $a_{r,n}$'s. We can see that, for $0 \leqq n \leqq r-2$, $(a_{r,n} \bmod r) \in \mathbb{Z}/r\mathbb{Z}$ is a topological invariant which is uniquely determined by $\tau'_r(M)$, since the ring $\mathbb{Z}[q]$ is isomorphic to $\mathbb{Z}[t]/T(t)$ where

$$\begin{aligned} T(t) &= 1 + t + t^2 + \cdots + t^{r-1} \\ &= \binom{r}{1} + \binom{r}{2}(t-1) + \binom{r}{3}(t-1)^2 + \cdots + \binom{r}{r}(t-1)^{r-1} \end{aligned}$$

noting that r is an odd prime.

Our aim is to show that there exist series of topological invariants $\lambda_n(M)$ of a rational homology 3-sphere M such that $(\frac{|H_1(M;\mathbb{Z})|}{r})\lambda_n(M)$ is congruent to $a_{r,n}$ modulo r for any odd prime r satisfying $r \geqq \max(2n+3, |H_1(M;\mathbb{Z})|)$, where $|H_1(M;\mathbb{Z})|$ is the order of $H_1(M,\mathbb{Z})$ and $(\frac{\bullet}{r})$ is the Legendre symbol. By our approach we can define $\lambda_n(M)$ for any closed oriented 3-manifold, but we can show that $\lambda_n(M)$ vanishes unless M is a rational homology 3-sphere. Our results for integral homology 3-spheres were already obtained in the previous paper [16].

In Sect. 1 we state main results. The following Sects. 2, 3 and 4 are devoted to proving the main theorem. In Sect. 5 we give the values of our invariant for lens spaces. Results in Sect. 5 suggests that our invariant $\tau(M)$ might describe the asymptotic behavior of quantum $SO(3)$ invariant $\tau'_r(M)$, see Example 1.6 and Remark 5.3.

1. Main results

Let M be a rational homology 3-sphere. Instead of quantum $SU(2)$ invariants $\tau_r(M)$ defined by Reshetikhin and Turaev [17], we use quantum $SO(3)$ invariants $\tau'_r(M)$ defined by Kirby and Melvin [9, Corollary 8.9], see Remark 2.2 below. H. Murakami [14] showed that $\tau'_r(M)$ belongs to $\mathbb{Z}[q]$ ($q = \exp(2\pi\sqrt{-1}/r)$) for any odd prime r and any rational homology 3-sphere M.

Let $\mathbb{Z}_{(m)}$ be $\mathbb{Z}[1/2, 1/3, \ldots, 1/m]$. We mean the order of $H_1(M;\mathbb{Z})$ by $|H_1(M;\mathbb{Z})|$.

The following theorem is our main theorem, which is proved in the following section.

Theorem 1.1. *Let M be a rational homology 3-sphere. Then there exist series of topological invariants of M, $\lambda_n = \lambda_n(M) \in \mathbb{Z}_{(\max\{|H_1(M;\mathbb{Z})|, 2n+1\})}$ for $n = 0, 1, 2, \ldots$ such that for any odd prime r which is greater than $|H_1(M;\mathbb{Z})|$ the following formula holds*;

$$\begin{aligned} \tau'_r(M) = &\left(\frac{|H_1(M;\mathbb{Z})|}{r}\right) \\ &\times \{\underline{\lambda_0} + \underline{\lambda_1}(q-1) + \underline{\lambda_2}(q-1)^2 + \cdots + \underline{\lambda_{(r-3)/2}}(q-1)^{(r-3)/2}\} \\ &+ u(q-1)^{(r-1)/2} \end{aligned} \tag{1.1}$$

with some $u \in \mathbb{Z}[q]$, *where* $(\frac{\bullet}{r})$ *is the Legendre symbol. Here* $\underline{\lambda_n}$ *is the image of* λ_n *by the following map*;

$$\mathbb{Z}_{(\max\{|H_1(M;\mathbb{Z})|, 2n+1\})} \xrightarrow{i} \mathbb{Z}_r \xrightarrow{p} \mathbb{Z}/r\mathbb{Z} \xrightarrow{=} \{0, 1, \dots, r-1\}$$

where $\mathbb{Z}_r$ *is the ring of* r*-adic integers and* i *is the natural inclusion and* p *is the projection, that is,* $\underline{\lambda_n}$ *is a reduction of* λ_n *modulo* r.

Note that the right-hand side of (1.1) is an expansion not in the ordinary topology but in r-adic topology. In fact, since u depends on r, the residue term $u(q-1)^{(r-1)/2}$ is not necessarily small, though it is small in r-adic topology; note that $(q-1)^{r-1}$ is divisible by r in $\mathbb{Z}[q]$. We will discuss about a relation between λ_n's and the asymptotic behavior of $\tau'_r(M)$ in the ordinary topology in Example 1.6 below. It would be necessary to take reduction modulo r in order to get coefficients in the expansion of the sequence $\tau'_r(M)$ which are independent of r, since the sequence splits into some converging subsequences in the ordinary topology, as shown in Example 1.6 below.

We also note that we expand $\tau'_r(M)$ as a power series not in h putting $h = 2\pi\sqrt{-1}/r$ but in $q-1$ because of technical request that $q-1$ is small in r-adic topology while h is big.

Definition 1.2. *We define a topological invariant* $\tau(M)$ *for any oriented rational homology* 3*-sphere* M *by*

$$\tau(M) = \sum_{n=0}^{\infty} \lambda_n (t-1)^n$$

which is formal power series in $t-1$ *with coefficients in* $\mathbb{Q}$.

Remark 1.3. In the same way as in this paper, we can define the above $\tau(M)$ for any closed oriented 3-manifold M. However one can prove that $\tau(M) = 0$ unless M is a rational homology 3-sphere.

Remark 1.4. By H. Murakami's results [14], we have

$$\lambda_0 = 1/|H_1(M;\mathbb{Z})| ,$$
$$\lambda_1 = 6\lambda(M)/|H_1(M;\mathbb{Z})| ,$$

where $\lambda(M)$ is Casson–Walker invariant [19]. Here we use Casson's normalization, which is half times Walker's definition.

We can immediately obtain the following proposition from properties of $\tau'_r(M)$ [9].

Proposition 1.5. $\tau(M)$ *satisfies the following properties*;

$$\tau(S^3) = 1$$
$$\tau(M_1 \# M_2) = \tau(M_1)\tau(M_2)$$
$$\tau(-M)(t) = \tau(M)(t^{-1})$$

where $M_1 \# M_2$ *is the connected sum of* M_1 *and* M_2, *and* $-M$ *is* M *with the opposite orientation.*

We might have more well-behaved coefficients instead of λ_n's if we expand $\tau(M)$ as power series in $\hbar$ putting $t = e^{\hbar} = 1 + \hbar + \frac{1}{2}\hbar^2 + \cdots \in \mathbb{Q}[[\hbar]]$, to describe difference between $\tau(M)$ and $\tau(-M)$.

Example 1.6. We see a relation between $\tau(M)$ and asymptotic behavior of $\tau'_r(M)$ in the case that M is a lens space $L(5,1)$. We have quantum $SO(3)$ invariant of $L(5,1)$ (see [5], [10]) as

$$\begin{aligned}\tau'_r(L(5,1)) &= \left(\frac{5}{r}\right) q^{-3\cdot\overline{5}} \frac{q^{\overline{10}} - q^{-\overline{10}}}{q^{\overline{2}} - q^{-\overline{2}}} \\ &= \left(\frac{5}{r}\right)(\overline{5} - 3\cdot\overline{5}^2(q-1) + 11\cdot\overline{5}^3(q-1)^2 + \cdots)\end{aligned}$$

where we mean the inverse of m in $\mathbb{Z}/r\mathbb{Z}$ by $\overline{m}$. By comparing the above formula with

$$t^{-3/5}\frac{t^{1/10} - t^{-1/10}}{t^{1/2} - t^{-1/2}} = \frac{1}{5} - \frac{3}{5^2}(t-1) + \frac{11}{5^3}(t-1)^2 + \cdots,$$

we obtain $\tau(M) = t^{-3/5}(t^{1/10} - t^{-1/10})/(t^{1/2} - t^{-1/2}) \in \mathbb{Q}[[t-1]]$ by Theorem 1.1.

Note that $q^{\overline{10}}$ is not necessarily equal to $q^{1/10}$. The sequence of $q^{\overline{10}}$ for $r = 7, 11, 13, \ldots$ splits into subsequences, which correspond to subsequences of r consisting of primes which are congruent to $1, 3, 7$ and 9 modulo 10. For a subsequence of primes r which are congruent to -1 modulo 10, we can put $\overline{10} = (r+1)/10$ and obtain $q^{\overline{10}} = \zeta q^{1/10}$ where we put $\zeta = \exp(2\pi\sqrt{-1}/10)$ and regard $\exp(2\pi\sqrt{-1}/10r)$ as $q^{1/10}$. Since $\left(\frac{5}{r}\right) = 1$ holds for these primes, we have asymptotic behavior of $\tau'_r(L(5,1))$ for these primes as

$$\begin{aligned}\tau'_r(L(5,1)) &= \zeta^{-6}e^{-3h/5}\frac{\zeta e^{h/10} - \zeta^{-1}e^{-h/10}}{\zeta^5 e^{h/2} - \zeta^{-5}e^{-h/2}} \\ &= \tau(L(5,1))\big|_{t^{1/10} = \zeta e^{h/10}}\end{aligned}$$

for primes r which are congruent to -1 modulo 10, where $h = 2\pi\sqrt{-1}/r$. Since λ_n's appear in the expansion of $\tau(L(5,1))$ at $t^{1/10} = e^{h/10}$, the above asymptotic behavior can not be described using finite $\lambda_n(L(5,1))$'s, though we can describe it using $\tau(L(5,1))$ at $t^{1/10} = \zeta e^{h/10} \in \mathbb{Q}[[t-1]]$.

If we put $q = \exp(10\pi\sqrt{-1}/r)$ in the definition of $\tau'_r((5,1))$ instead of putting $q = \exp(2\pi\sqrt{-1}/r)$, we have $q^{\overline{10}} = (q^{\overline{5}})^{\overline{2}} = \exp(2\pi\sqrt{-1}/r)^{(r+1)/2} = -e^{h/10}$ where $h = 10\pi\sqrt{-1}/r$. Hence we have

$$\begin{aligned}\tau'_r(L(5,1))\big|_{q=e^h=\exp(10\pi\sqrt{-1}/r)} &= e^{-3h/5}\frac{e^{h/10} - e^{-h/10}}{e^{h/2} - e^{-h/2}} \\ &= \tau(L(5,1))\big|_{t=e^h} \\ &= (\lambda_0 + \lambda_1(t-1) + \lambda_2(t-1)^2 + \cdots)\big|_{t=e^h} \\ &= \lambda_0 + \lambda_1 h + \left(\frac{\lambda_1}{2} + \lambda_2\right)h^2 + \cdots\end{aligned}$$

Therefore we can describe the asymptotic behavior of this series of quantum $SO(3)$ invariants using λ_n's.

In general we could expect that the sequence of $\tau_r'(M)$ splits into subsequences whose asymptotic behavior can be described as $\tau(M)|_{t^{1/m}=\zeta e^{h/m}}$ where m is an integer somehow related to $|H_1(M;\mathbb{Z})|$ and ζ is an m-th root of unity. We could also expect that the first n coefficients in the expansion of the series $\tau_r'(M)|_{q=e^h}$ (where $h = 2m'\pi\sqrt{-1}/r$ with some m' somehow related to $|H_1(M;\mathbb{Z})|$) are linearly related to $\lambda_0, \ldots, \lambda_{n-1}$, through a coordinate change between a power series in $q-1$ and a power series in h. These two observations are true for every lens space $L(\alpha,\beta)$ putting $m = 2\alpha$ and $m' = \alpha$, see Sect. 5.

Notations. In this paper we use the same notations as in [16].

Throughout this paper r is an odd prime. We set $q = \exp(2\pi\sqrt{-1}/r)$ and $q^{1/2} = \exp(\pi\sqrt{-1}/r)$. We also put $[k] = (q^{k/2} - q^{-k/2})/(q^{1/2} - q^{-1/2})$. We denote the inverse of $m \in \mathbb{Z}/r\mathbb{Z}$ in $\mathbb{Z}/r\mathbb{Z}$ by $\bar{m} \in \mathbb{Z}/r\mathbb{Z}$. For non-zero integer f we put sign $(f) = f/|f|$. We denote by $(\frac{\bullet}{r})$ the Legendre symbol. We put $G(q)$ to be Gauss sum; $\sum_{k=0}^{r-1} q^{k^2}$. We set $\mathbb{Z}_{(m)} = \mathbb{Z}[1/2, 1/3, \ldots, 1/m]$.

We express a higher term in power expansion in $q-1$ in the following way. We denote

$$\cdots + u(q-1)^k \quad \text{with some } u \in R[q]$$

by

$$\cdots + O((q-1)^k; R)$$

where R is $\mathbb{Z}$ or $\mathbb{Z}_{(r-1)}$.

We use bold faces (e.g. $\mathbf{k}$) for multi-indices. For $\mathbf{k} = (k_1, k_2, \ldots, k_\mu)$, $\mathbf{j} = (j_1, j_2, \ldots, j_\mu)$ and a scalar x, we put

$$|\mathbf{k}| = k_1 + k_2 + \cdots + k_\mu\ ,$$

$$\#\mathbf{k} = \mu\ ,$$

$$\mathbf{k}! = k_1! k_2! \ldots k_\mu!\ ,$$

$$[\mathbf{k}] = [k_1][k_2] \ldots [k_\mu]\ ,$$

$$\mathbf{k} + \mathbf{j} = (k_1 + j_1, k_2 + j_2, \ldots, k_\mu + j_\mu)\ ,$$

$$\mathbf{k}\ \mathbf{j} = (k_1 j_1, k_2 j_2, \ldots, k_\mu j_\mu)\ ,$$

$$\begin{pmatrix} \mathbf{k} \\ \mathbf{j} \end{pmatrix} = \begin{pmatrix} k_1 \\ j_1 \end{pmatrix} \begin{pmatrix} k_2 \\ j_2 \end{pmatrix} \cdots \begin{pmatrix} k_\mu \\ j_\mu \end{pmatrix}\ ,$$

$$x\mathbf{k} = (xk_1, xk_2, \ldots, xk_\mu)\ ,$$

$$x^{\mathbf{k}} = x^{k_1} x^{k_2} \ldots x^{k_\mu}\ ,$$

$$\mathbf{x} = (x, x, \ldots, x)\ .$$

We mean $k_1 > j_1, k_2 > j_2, \dots, k_\mu > j_\mu$ by $\mathbf{k} > \mathbf{j}$ and $k_1 > x, k_2 > x, \dots, k_\mu > x$ by $\mathbf{k} > x$.

Let $L = L_1 \cup L_2 \cup \cdots L_\mu$ be an oriented link in S^3, and let $\mathscr{L} = (L, \mathbf{f})$ be a framed link with framing $\mathbf{f} = (f_1, f_2, \dots, f_\mu)$.

We denote by $\#L$ the number of components of L. We put $\sigma(L)$ to be the signature of the linking matrix of $\mathscr{L}$. We put $\tilde{\sigma}(\mathscr{L})$ to be the number of positive eigenvalues of the linking matrix of $\mathscr{L}$ or $\#L$ according to whether $r \equiv 1$ or $-1 \bmod 4$.

For a multi-index $\mathbf{m} = (m_1, m_2, \dots, m_\mu)$ we denote by $L^{\mathbf{m}}$ the $\mathbf{m}$-parallel of L with $\mathbf{0}$-framing; it is a $|\mathbf{m}|$ component link consisting of a union of m_ξ parallel copies of L_ξ and a linking number of any two components among the same m_ξ copies is zero. In particular we define L^0 to be empty link.

Let $V(L;t)$ be the Jones polynomial [7]. We put

$$X(L;t) = V(L;t)/(t^{1/2} + t^{-1/2})^{\#L-1} ,$$

noting that $X(L;t)$ belongs to $\mathbb{Z}[t, t^{-1}, 1/(t+1)]$ and $X(L;q)$ belongs to $\mathbb{Z}[q]$ since $q+1$ is invertible in $\mathbb{Z}[q]$. We define $X(\text{empty link};\, t)$ to be 1. We put

$$X^{(d)}(L) = \left(\frac{d}{dt}\right)^d X(L;t)|_{t=1} .$$

Moreover we put

$$\Phi(L;t) = (-1)^{\#L} + \sum_{L' \subset L} (-1)^{\#L - \#L'} X(L';t)$$

where the sum runs over all sublinks of L except empty link. (The first term corresponds to a contribution from empty link.) We define Φ (empty link; t) to be 0. We also put

$$\Phi^{(d)}(L) = \left(\frac{d}{dt}\right)^d \Phi(L;t)\bigg|_{t=1} .$$

Note that $X^{(d)}(L)/d!$ and $\Phi^{(d)}(L)/d!$ belong to $\mathbb{Z}[1/2]$ since they are coefficients of Taylor expansion of $X(L;t)$ and $\Phi(L;t)$ at $t = 1$ respectively.

2. Proof of Theorem 1.1

We begin with the definition of quantum $SO(3)$ invariants $\tau_r'(M)$.

Let $L = L_1 \cup \cdots \cup L_\mu$ be an oriented link in S^3. We call L algebraically split if $\mathrm{lk}(L_\xi, L_\nu) = 0$ for any ξ and ν where $\mathrm{lk}(L_\xi, L_\nu)$ is the linking number of L_ξ and L_ν. A *framed link* $\mathscr{L} = (L, \mathbf{f})$ is a link L with framing $\mathbf{f} = (f_1, f_2, \dots, f_\mu)$, $f_\xi \in \mathbb{Z}$.

Let M be a closed oriented 3-manifold obtained by integral surgery along a framed link $\mathscr{L} = (L, \mathbf{f})$.

Theorem and Definition 2.1 [9, Corollary 8.9]. *For any odd r, put $\tau'_r(M)$ to be*

$$\left(\frac{2}{\sqrt{r}}\sin\frac{\pi}{r}\right)^{\mu}\exp\left(2\pi\sqrt{-1}\cdot\frac{\pm\sigma(\mathscr{L})}{8}\right)\exp\left(-2\pi\sqrt{-1}\cdot\frac{3(r-2)}{8r}\cdot\sigma(\mathscr{L})\right)$$
$$\times\sum_{\mathbf{k}=1}^{(r-1)/2}\sqrt{-1}^{r\,{}^tE_{\mathbf{k}}AE_{\mathbf{k}}}[\mathbf{k}]J_{\mathscr{L},\mathbf{k}} \tag{2.1}$$

where the $+$ or $-$ sign is chosen according to whether $r\equiv 3$ or $r\equiv 1 \bmod 4$ and μ is the number of components in $\mathscr{L}$, A is the linking matrix of $\mathscr{L}$, $\sigma(\mathscr{L})$ is the signature of A. $E_{\mathbf{k}}=(E_1(k_1),\,E_2(k_2),\dots,E_\mu(k_\mu))$ with $E_\xi(k_\xi)=1$ if k_ξ is even and 0 otherwise, and $J_{\mathscr{L},\mathbf{k}}$ is the colored framed link invariant (see [9]). Then $\tau'_r(M)$ does not depend on a choice of $\mathscr{L}$ and becomes a topological invariant of a closed oriented 3-manifold M. We call $\tau'_r(M)$ a quantum $SO(3)$ invariant of M.

Remark 2.2. As in [9], $\tau'_r(M)$ satisfies the following property for odd r;

$$\tau_r(M)=\begin{cases}\overline{\tau_3(M)}\;\tau'_r(M) & \text{if } r\equiv 1 \bmod 4\\ \tau_3(M)\;\tau'_r(M) & \text{if } r\equiv -1 \bmod 4\end{cases}$$

where $\tau_r(M)$ is the quantum $SU(2)$ invariant. Note that $\tau_3(M)=1$ for any integral homology 3-sphere M, and so Theorem 1.1 is a generalization of the corresponding result in [16].

In this paper we use $\tau'_r(M)$ instead of $\tau_r(M)$ by the following three reasons. Firstly $\tau_r(M)$ is easily obtained from $\tau'_r(M)$ since $\tau_3(M)$ is a well-known invariant. In fact $\tau_3(M)$ depends only on cohomology ring structure of M, see [15]. Note that $\tau'_r(M)$ is not necessarily obtained from $\tau_r(M)$ since $\tau_3(M)$ sometimes vanishes. Secondly $\tau'_r(M)$ is more well-behaved than $\tau_r(M)$ from our viewpoint as in Remark 1.4. For example, $\tau'_r(M)$ always belongs to $\mathbb{Z}[q]$, but $\tau_r(M)$ does not necessarily so. Thirdly $\tau'_r(M)$ might be more suitable when we investigate the asymptotic formula of quantum invariants, see Remark 5.3.

Under the assumption that r is odd prime. L is algebraically split and M is a rational homology 3-sphere, H. Murakami [14] modified (2.1), to obtain the following formula;

$$\tau'_r(M)=(-1)^{\tilde\sigma(\mathscr{L})}q^{3\cdot\bar4\sigma(\mathscr{L})-\bar2\mu}\left(\frac{q-1}{G(q)}\right)^{\mu}\sum_{\mathbf{k}=1}^{(r-1)/2}q^{\bar4\mathbf{f}(\mathbf{k}^2-1)}[\mathbf{k}]$$
$$\times\sum_{\mathbf{j}=0}^{(\mathbf{k}-1)/2}(-1)^{\mathbf{j}}\binom{\mathbf{k}-\mathbf{j}-\mathbf{1}}{\mathbf{j}}[2]^{\mathbf{k}-2\mathbf{j}-\mathbf{1}}X(L^{\mathbf{k}-2\mathbf{j}-\mathbf{1}};q) \tag{2.2}$$

where $\tilde\sigma(\mathscr{L})$ is defined in Sect. 1. In order to use this formula we must assume that M is obtained by integral surgery along an algebraically split framed link. Hence we must reduce the case of general M to the case of such M.

Now we reduce Theorem 1.1 to the following proposition, which is proved using (2.2) in the following section.

Proposition 2.3. *Let M be a rational homology* 3*-sphere obtained by integral surgery along an algebraically split framed link $\mathscr{L}$. Then there exist $\lambda_n = \lambda_n(M, \mathscr{L}) \in \mathbb{Z}_{(\max\{|H_1(M;\mathbf{Z})|, 2n+1\})}$ for $n = 0, 1, 2, \dots$ such that for any odd prime r which does not divide $|H_1(M;\mathbb{Z})|$ the following formula holds*;

$$\tau'_r(M) = \left(\frac{|H_1(M;\mathbb{Z})|}{r}\right)$$
$$\times \{\lambda_0 + \lambda_1(q-1) + \lambda_2(q-1)^2 + \cdots + \lambda_{(r-3)/2}(q-1)^{(r-3)/2}\}$$
$$+ O((q-1)^{(r-1)/2}; \mathbb{Z}_{(r-1)}) . \tag{2.3}$$

Proof of Theorem 1.1 *assuming Proposition* 2.3.
Case 1. If M can be obtained by integral surgery along some algebraically split framed link $\mathscr{L}$, by Proposition 2.3 we immediately obtain (1.1) for $\lambda_n = \lambda_n(M, \mathscr{L})$, noting that $\tau'_r(M)$ belongs to $\mathbb{Z}[q]$ (see [14]). By topological invariance of $\tau'_r(M)$, $\underline{\lambda_n}$ is a topological invariant of M for infinitely many r. Hence each λ_n is a topological invariant of M, i.e. does not depend on a choice of $\mathscr{L}$. Therefore we obtain Theorem 1.1 in this case.
Case 2. If M can not be obtained by integral surgery along any algebraically split framed link, we can reduce this case to Case 1 by obtaining (2.3) as follows.

By Corollary 2.5 below, there exist lens spaces $L(n_1, 1), L(n_2, 1), \dots, L(n_v, 1)$ with $|n_\xi| \leqq |H_1(M;\mathbb{Z})|$ such that $M' = M \# L(n_1, 1) \# \cdots \# L(n_v, 1)$ can be obtained by integral surgery along some algebraically split framed link $\mathscr{L}$. Since the quantum $SO(3)$ invariant is multiplicative under connected sum [9], we have

$$\tau'_r(M') = \tau'_r(M) \cdot \prod_{\xi=1}^{v} \tau'_r(L(n_\xi, 1)) . \tag{2.4}$$

Applying Proposition 2.3 to M' and the lens spaces, we obtain topological invariants $\lambda_n(M') \in \mathbb{Z}_{(\max\{|H_1(M',\mathbf{Z})|, 2n+1\})}$ and $\lambda_n(L(n_\xi, 1)) \in \mathbb{Z}_{(2n+1)}[1/n_\xi]$ for $n = 0, 1, 2, \dots$. Noting that $n_\xi \cdot \lambda_0(L(n_\xi, 1)) = 1$ (see Remark 1.4), $n_\xi \cdot \tau'_r(L(n_\xi, 1))$ is invertible in $\mathbb{Q}[[q-1]]$ and the n-th coefficient of the inverse belongs to $\mathbb{Z}_{(2n+1)}[1/n_\xi]$. Since $|H_1(M', \mathbb{Z})| = |H_1(M;\mathbb{Z})| \cdot \prod n_\xi$ holds and the Legendre symbol is multiplicative under product, by (2.4) we have (2.3) for M with series of $\lambda_n(M) \in \mathbb{Z}_{(\max\{|H_1(M;\mathbf{Z})|, 2n+1\})}$. By topological invariance of $\tau'_r(M), \lambda_n(M)$ is a topological invariant of M, i.e. does not depend on a choice of a set of lens spaces. Hence we obtain Theorem 1.1 in this case, completing the proof. □

In order to diagonalize a linking matrix we need the following lemma, whose proof will be given in Sect. 4.

Lemma 2.4. *Let A be a symmetric non-singular integral matrix. Then there exist integers $n_1, n_2, \dots, n_v$ with $|n_\xi| \leqq |\det A|$ and a unimodular matrix P*

such that

$$ {}^tP \cdot (A \oplus (n_1) \oplus (n_2) \oplus \cdots \oplus (n_v)) \cdot P $$

is a diagonal matrix. Here we mean the block sum by $\oplus$.

Let M be any closed oriented 3-manifold, and let $\mathscr{L}$ be a framed link such that M is obtained by integral surgery along $\mathscr{L}$. By applying the above lemma to the linking matrix of $\mathscr{L}$, we obtain the following corollary.

Corollary 2.5. *For any closed oriented* 3*-manifold M, there exist lens spaces* $L(n_1, 1), L(n_2, 1), \ldots, L(n_v, 1)$ *with* $n_\xi \leqq |H_1(M; \mathbb{Z})|$ *such that* $M \# L(n_1, 1) \# L(n_2, 1) \# \cdots \# L(n_v, 1)$ *can be obtained by integral surgery along some algebraically split framed link.*

3. Proof of Proposition 2.3

This section is devoted to proving Proposition 2.3. We prove it modifying the proof in [16] which dealt with integral homology 3-spheres. In this section we will proceed as follows. Since we want the expansion of $\tau_r'(M)$ in $q-1$, we must expand $J_{L,\mathbf{k}}$ in (2.1), which can be replaced with Jones polynomial as in (2.2); recall $X(L; t)$ is normalized Jones polynomial such that $X(\text{trivial link}; t) = 1$. The most difficulty we have is that the range of the sum in (2.2) depends on r while we want to get coefficients which are independent of r in the expansion of $\tau_r'(M)$ in $q-1$. The key to avoid the difficulty is to replace $X(L; t)$ with $\Phi(L; t)$ whose coefficients in the expansion at $t = 1$ vanish for sufficiently large parallel L of a link, where the sum runs over parallels of a link. Proposition 3.1 replaces $X(L)$ with $\Phi(L)$, and Proposition 3.2 shows the vanishing of the coefficients of $\Phi(L)$. Proposition 3.3 shows that the last line in (3.2) will be divisible sufficiently many times by $q-1$ when it is substituted in (2.2). By using these three propositions we can express $\tau_r'(M)$ as a sum whose range concerning L does not depend on r as in (3.3). Proposition 3.4 expands the rest part not concerning L in (3.3) into a power series in $q-1$; it is merely numerical calculation.

We fix arbitrary positive integer n and suppose $r \geqq 2n+3$. We replace X in (2.2) as follows. Take Taylor expansion of $X(L^{\mathbf{k}-2\mathbf{j}-1}; t)$ at $t = 1$ and put $t = q$, then we have

$$ X(L^{\mathbf{k}-2\mathbf{j}-\mathbf{1}}; q) = \sum_{d=0}^{\mu(r-3)/2+n} \frac{X^{(d)}(L^{\mathbf{k}-2\mathbf{j}-\mathbf{1}})}{d!}(q-1)^d $$

$$ + O\left((q-1)^{\mu(r-3)/2+n+1}; \mathbb{Z}\left[\frac{1}{2}\right]\right) \tag{3.1} $$

where $\mu = \# L$. Now we have the following three proposition.

Proposition 3.1 ([16]). *Let L be an algebraically split link and let n be any positive integer. Then the following formula holds.*

$$X^{(d)}(L)=\begin{cases}1 & \text{if } d=0\\ 0 & \text{if } d=1\\ \sum_{l=1}^{d-1}\sum_{\substack{\#L'=d-l\\ L'\subset L}}\Phi^{(d)}(L') & \text{if } 2\leqq d\leqq n+1\\ \sum_{l=1}^{n}\sum_{\substack{\#L'=d-l\\ L'\subset L}}\Phi^{(d)}(L') & \\ -(-1)^{d+n}\sum_{v=1}^{d-n-1}(-1)^{v}\binom{\#L-v-1}{d-v-n-1}\sum_{\substack{\#L'=v\\ L'\subset L}}X^{(d)}(L') & \text{if } d\geqq n+2\end{cases}\tag{3.2}$$

Proposition 3.2 ([16]). *Suppose L is an algebraically split link and contains m-parallel (i.e. there exists an algebraically split link $\hat{L}$ such that L is obtained from $\hat{L}$ by taking m-parallel with 0-framing on some component of $\hat{L}$). Then we have $\Phi^{(d)}(L)=0$ if $0\leqq d\leqq \#L+m-1$.*

Proposition 3.3 ([13, 16]). *For any d, v and $\mathbf{i}$ satisfying $1\leqq v=|\mathbf{i}|\leqq d-n-1$, we have*

$$\begin{aligned}&\sum_{\mathbf{k}=1}^{(r-1)/2}q^{\bar{4}\mathbf{f}(\mathbf{k}^2-\mathbf{1})}[\mathbf{k}]\sum_{\mathbf{j}=0}^{(\mathbf{k}-\mathbf{1})/2}(-1)^{\mathbf{j}}\begin{pmatrix}\mathbf{k}-\mathbf{j}-\mathbf{1}\\ \mathbf{j}\end{pmatrix}[2]^{\mathbf{k}-2\mathbf{j}-\mathbf{1}}\\ &\quad\times\begin{pmatrix}|\mathbf{k}-2\mathbf{j}-\mathbf{1}|-v-1\\ d-v-n-1\end{pmatrix}\begin{pmatrix}\mathbf{k}-2\mathbf{j}-\mathbf{1}\\ \mathbf{i}\end{pmatrix}\\ &=O((q-1)^{\mu(r-3)/2-d+n+1};\mathbb{Z})\,.\end{aligned}$$

We substitute (3.1) to (2.2), and replace $X^{(d)}$ with $\Phi^{(d)}$ by Proposition 3.1. Then some terms vanish by Proposition 3.2, and other terms become remainder terms by Proposition 3.3. (For concrete calculations, see [16].) Hence we have the following formula.

$$\begin{aligned}\tau_r'(M)&=(-1)^{\bar{\sigma}(\mathscr{L})}q^{3\cdot\bar{4}\sigma(\mathscr{L})-\bar{2}\mu}\left(\frac{q-1}{G(q)}\right)^{\mu}\\ &\quad\times\sum_{\mathbf{k}=1}^{(r-1)/2}q^{\bar{4}\mathbf{f}(\mathbf{k}^2-\mathbf{1})}[\mathbf{k}]\sum_{\mathbf{j}=0}^{(\mathbf{k}-\mathbf{1})/2}(-1)^{\mathbf{j}}\begin{pmatrix}\mathbf{k}-\mathbf{j}-\mathbf{1}\\ \mathbf{j}\end{pmatrix}[2]^{\mathbf{k}-2\mathbf{j}-\mathbf{1}}\\ &\quad\times\left(1+\sum_{l=1}^{n}\sum_{\mathbf{i}=0}^{l}\begin{pmatrix}\mathbf{k}-2\mathbf{j}-\mathbf{1}\\ \mathbf{i}\end{pmatrix}\frac{\Phi^{(l+|\mathbf{i}|)}(L^{\mathbf{i}})}{(l+|\mathbf{i}|)!}(q-1)^{l+|\mathbf{i}|}\right)\\ &\quad+O\left((q-1)^{n+1};\mathbb{Z}\left[\frac{1}{2}\right]\right)\\ &=\prod_{\xi=1}^{\mu}\left(-\mathrm{sign}(f_{\xi})\left(\frac{\mathrm{sign}(f_{\xi})}{r}\right)q^{3\cdot\bar{4}\,\mathrm{sign}(f_{\xi})-\bar{2}}\frac{q-1}{G(q)}\right.\end{aligned}$$

$$\times \sum_{k_\xi=1}^{(r-1)/2} q^{\bar{4} f_\xi (k_\xi^2-1)} [k_\xi] \sum_{j_\xi=0}^{(k_\xi-1)/2} (-1)^{j_\xi}$$

$$\times \left(\begin{matrix} k_\xi - j_\xi - 1 \\ j_\xi \end{matrix} \right) [2]^{k_\xi - 2j_\xi - 1} \Bigg) + q^{3 \cdot \bar{4}\sigma(\mathscr{L}) - \bar{2}\mu} \sum_{l=1}^{n} \sum_{\mathbf{i}=0}^{l} \frac{\Phi^{(l+|\mathbf{i}|)}(L^{\mathbf{i}})}{(l+|\mathbf{i}|)!} (q-1)^l$$

$$\times \prod_{\xi=1}^{\mu} \left(-\mathrm{sign}(f_\xi) \left(\frac{\mathrm{sign}(f_\xi)}{r} \right) \frac{(q-1)^{i_\xi+1}}{G(q)} \sum_{k_\xi=1}^{(r-1)/2} q^{\bar{4} f_\xi (k_\xi^2 - 1)} [k_\xi] \right.$$

$$\left. \times \sum_{j_\xi=0}^{(k_\xi-1)/2} (-1)^{j_\xi} \left(\begin{matrix} k_\xi - j_\xi - 1 \\ j_\xi \end{matrix} \right) \left(\begin{matrix} k_\xi - 2j_\xi - 1 \\ i_\xi \end{matrix} \right) [2]^{k_\xi - 2j_\xi - 1} \right)$$

$$+ O\left((q-1)^{n+1}; \mathbf{Z}\left[\frac{1}{2}\right] \right) . \tag{3.3}$$

Since $\prod \mathrm{sign}(f_\xi) \cdot \prod f_\xi = \prod |f_\xi| = |H_1(M;\mathbf{Z})|$, we use Proposition 3.4 below to obtain the following formula.

$$\tau'_r(M) = \left(\frac{|H_1(M;\mathbf{Z})|}{r} \right) \prod_{\xi=1}^{\mu} \left(-\mathrm{sign}(f_\xi) q^{\bar{2} - \bar{4} f_\xi + 3 \cdot \bar{4}\,\mathrm{sign}(f_\xi)} \cdot \frac{q^{-\bar{f}_\xi} - 1}{q-1} \right)$$

$$+ \left(\frac{|H_1(M;\mathbf{Z})|}{r} \right) q^{3 \cdot \bar{4}\sigma(\mathscr{L}) - \bar{2}\mu} \sum_{l=1}^{n} \sum_{\mathbf{i}=0}^{l} \frac{\Phi^{(l+|\mathbf{i}|)}(L^{\mathbf{i}})}{(l+|\mathbf{i}|)!} (q-1)^l$$

$$\times \prod_{\xi=1}^{\mu} \left(-\mathrm{sign}(f_\xi) \sum_{m_\xi=0}^{n-l} h_{f_\xi, i_\xi, m_\xi} (q-1)^{m_\xi} \right)$$

$$+ O\left((q-1)^{n+1}; \mathbf{Z}_{(r-1)}\left[\frac{1}{|H_1(M;\mathbf{Z})|} \right] \right) .$$

Hence we reformed the formula of $\tau'_r(M)$ $(r \geqq 2n+3)$ to a formula consisting of terms which are independent of r and higher terms. We can check that the coefficient of $(q-1)^n$ belongs to $\mathbf{Z}_{(\max\{|H_1(M;\mathbf{Z})|\ 2n+1\})}$. This completes the proof of Proposition 2.3. □

Proposition 3.4. *For each non-zero integer f and each non-negative integer m and i there exists $h_{f,i,m} \in \mathbf{Z}_{(2i+2m+1)}[1/f]$ such that the following formula holds for any f, i and any odd prime r which is not a prime factor of f and satisfies $r \geqq 2i+3$.*

$$\sum_{k=1}^{(r-1)/2} q^{\bar{4} f (k^2-1)} [k] \sum_{j=0}^{(k-1)/2} (-1)^j \left(\begin{matrix} k-j-1 \\ j \end{matrix} \right) \left(\begin{matrix} k-2j-1 \\ i \end{matrix} \right)$$

$$\times [2]^{k-2j-1} \times \frac{(q-1)^{i+1}}{G(q)}$$

$$= \left(\frac{f}{r} \right) \sum_{m=0}^{(r-1)/2-i-1} h_{f,i,m} (q-1)^m + O\left((q-1)^{(r-1)/2-i}; \mathbf{Z}_{(r-1)}\left[\frac{1}{f} \right] \right)$$

In particular for $i = 0$ *the left-hand side of the above formula is equal to*

$$\left(\frac{f}{r}\right) q^{1-\bar{4}f} \cdot \frac{q^{-\bar{f}} - 1}{q - 1}$$

Proof. We can easily show

$$[k] = \sum_{j=0}^{(k-1)/2} (-1)^j \binom{k-j-1}{j} [2]^{k-2j-1}$$

by induction on k. Then, in the case $i = 0$, we have

$$\begin{aligned}
\mathrm{LHS} &= \sum_{k=1}^{(r-1)/2} q^{\bar{4}f(k^2-1)}[k]^2 \times \frac{q-1}{G(q)} \\
&= \frac{q^{1-\bar{4}f}}{(q-1)G(q)} \sum_{k=1}^{(r-1)/2} (q^{\bar{4}fk^2+k} + q^{\bar{4}fk^2-k} - 2q^{\bar{4}fk^2}) \\
&\qquad \left(\text{recall } [k] = \frac{q^{k/2} - q^{-k/2}}{q^{1/2} - q^{-1/2}}\right) \\
&= \frac{q^{1-\bar{4}f}}{(q-1)G(q)} \sum_{k=1}^{r-1} (q^{\bar{4}fk^2+k} - q^{\bar{4}fk^2}) \\
&\qquad (\text{replacing } k \text{ by } r-k \text{ in some terms}) \\
&= \frac{q^{1-\bar{4}f}}{(q-1)G(q)} \sum_{k=0}^{r-1} (q^{\bar{4}fk^2-\bar{f}} - q^{\bar{4}fk^2}) \\
&\qquad (\text{replacing } k \text{ by } k-2\bar{f} \text{ in the first term}) \\
&= \left(\frac{4f}{r}\right) q^{1-\bar{4}f} \cdot \frac{q^{-\bar{f}} - 1}{q - 1}
\end{aligned}$$

Since $(\frac{4f}{r}) = (\frac{f}{r})$, we obtain the required formula.

In the case $i \geqq 1$ we can obtain the required result in a similar way as in [16]. □

4. Proof of Lemma 2.4

In this section we prove Lemma 2.4. For a similar proof for $\mathbb{Z}/r\mathbb{Z}$-homology spheres, see [14].

Let G be a finite Abelian group. A *linking pairing* on G is a non-singular symmetric bilinear map of $G \times G$ to $\mathbb{Q}/\mathbb{Z}$. For a non-singular symmetric integral $n \times n$ matrix A, we have an induced linking pairing ϕ on $\mathbb{Z}^n/A\mathbb{Z}^n$ defined by $\phi([v],[v']) = {}^t v A^{-1} v'$ for $v, v' \in \mathbb{Z}^n$ whose images in $\mathbb{Z}^n/A\mathbb{Z}^n$ are denoted by $[v],[v']$; note that the right-hand side of this formula is well-defined in $\mathbb{Q}/\mathbb{Z}$.

We denote this linking pairing by $\imath(A)$. It is known [11, 3] that if two non-singular symmetric integral matrices A_1, A_2 give the same linking pairing $\imath(A_1)$ and $\imath(A_2)$ then there exists a unimodular integral matrix P such that

$$ {}^tP \cdot (A_1 \oplus (\pm 1) \oplus \cdots \oplus (\pm 1)) \cdot P = A_2 \oplus (\pm 1) \oplus \cdots \oplus (\pm 1) . $$

The set of linking pairing becomes an Abelian semigroup with respect to direct sum. Generators and relations of the semigroup are known [20, 8]. The generators in [20] are:

$[1/p^k], [d_p/p^k]$ on $\mathbb{Z}/p^k\mathbb{Z}$

for p odd primes, d_p a quadratic non-residue modulo p

$[1/2]$ on $\mathbb{Z}/2\mathbb{Z}$, $[1/2^2], [-1/2^2]$ on $\mathbb{Z}/2^2\mathbb{Z}$

$[1/2^k], [-1/2^k], [3/2^k], [-3/2^k]$ on $\mathbb{Z}/2^k\mathbb{Z}$ for $k \geqq 3$

E_0^k on $\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}$ for $k \geqq 1$

E_1^k on $\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}$ for $k \geqq 2$

where we denote by $[a/b]$ a linking pairing ϕ on $\mathbb{Z}/b\mathbb{Z}$ defined by $\phi([v],[v']) = avv'/b$ for $v, v' \in \mathbb{Z}$ and we define E_0^k and E_1^k by

$$ E_0^k([v],[v']) = {}^tv \begin{pmatrix} 0 & 2^{-k} \\ 2^{-k} & 0 \end{pmatrix} v', \quad E_1^k([v],[v']) = {}^tv \begin{pmatrix} 2^{1-k} & 2^{-k} \\ 2^{-k} & 2^{1-k} \end{pmatrix} v' $$

for $v, v' \in \mathbb{Z} \oplus \mathbb{Z}$.

Proof of Lemma 2.4. By the above observation, we can reduce Lemma 2.4 to the following lemma, which means $A \oplus (\bigoplus_\xi (n_\xi)) \oplus (\oplus(\pm 1))$ can be diagonalized to a matrix with diagonal entries $m_1, \dots, m_\mu, \pm 1, \dots, \pm 1$, putting $\phi = \imath(A)$. □

Lemma 4.1. *Let ϕ be a linking pairing on a finite Abelian group G. Then there exists integers $n_1, n_2, \dots, n_\nu$ and $m_1, m_2, \dots, m_\mu$ such that*

$$ \phi \oplus \imath\left(\bigoplus_{\xi=1}^{\nu}(n_\xi)\right) = \imath\left(\bigoplus_{\eta=1}^{\mu}(m_\eta)\right) $$

and $|n_\xi| \leqq |G|$ hold where we mean the order of G by $|G|$.

Proof. We will show that ϕ can be deformed to a form $\imath(\bigoplus_{\xi=1}^{\nu}(n_\xi)) \ominus \imath(\bigoplus_{\eta=1}^{\mu}(m_\eta))$ where we mean removal of direct summand by $\ominus$. We will proceed by induction on the greatest prime factor of $|G|$.

Since ϕ is equal to a direct sum of some generators given above, it suffices to show this lemma when ϕ is equal to each generator.

If $\phi = [1/p^k]$, then we have $\phi = \imath((p^k))$.

If $\phi = [d_p/p^k]$, then we have

$$\begin{aligned}\phi &= 2[1/p^k] \ominus [d_p/p^k] \\ &= \iota(2(p^k)) \oplus [p^k/d_p] \ominus \iota((p^k d_p))\end{aligned}$$

using a relation $2[1/p^k] = 2[d_p/p^k]$ in [20] and a relation $\iota((ab)) = [a/b] \oplus [b/a]$ (for coprime integers a and b) which can be obtained by easy calculation. By the hypothesis of induction we can deform $[p^k/d_p]$ to a required form, completing this case.

If $\phi = [\pm 1/2^k]$, then we have $\phi = \iota((\pm 2^k))$.

If $\phi = [\pm 3/2^k]$ (for $k \geqq 3$), then we have

$$\begin{aligned}\phi &= 2[\mp 1/2^k] \ominus [\pm 3/2^k] \\ &= \iota(2(\mp 2^k)) \ominus (\iota((\pm 3 \cdot 2^k)) \ominus [\pm 2^k/3]) \\ &= \iota(2(\mp 2^k) \oplus (\pm(-1)^k \cdot 3)) \ominus \iota((\pm 3 \cdot 2^k))\end{aligned}$$

using a relation $2[\pm 3/2^k] = 2[\mp 1/2^k]$ in [8] and $[\pm 2^k/3] = [\pm(-1)^k/3]$, completing this case.

If $\phi = E_0^k$ or E_1^k, then we have

$$\begin{aligned}E_0^k &= ([1/2^k] \oplus [-1/2^k]) \ominus [-1/2^k] \\ &= \iota((2^k) \oplus 2(-2^k)) \ominus \iota((-2^k)) \\ E_1^k &= 3[1/2^k] \ominus [3/2^k] \\ &= \iota(3(2^k) \oplus ((-1)^k \cdot 3)) \ominus \iota((3 \cdot 2^k))\end{aligned}$$

using relations in [8], completing the proof. □

5. Values of lens spaces

Example 5.1. Let α be an odd positive integer, β a coprime positive integer, then the value of lens space $L(\alpha, \beta)$ is as follows;

$$\tau(L(\alpha,\beta)) = t^{-3 \cdot s(\beta,\alpha)} \cdot \frac{t^{1/2\alpha} - t^{-1/2\alpha}}{t^{1/2} - t^{-1/2}}$$

where $s(\beta, \alpha)$ is Dedekind sum, that is,

$$s(\beta,\alpha) = \sum_{k=1}^{\alpha-1} \frac{k}{\alpha}\left(\frac{k\beta}{\alpha} - \left[\frac{k\beta}{\alpha}\right] - \frac{1}{2}\right).$$

The above formula of $\tau(L(\alpha,\beta))$ follows from the following proposition in the same way as in Example 1.6.

Proposition 5.2 ([10, 5]). *If r is an odd prime and α is odd and not divisible by r, then the value of quantum $SO(3)$ invariant of lens space is given by the* *following formula*;

$$\tau_r'(L(\alpha,\beta)) = \left(\frac{\alpha}{r}\right) q^{-(3\cdot s(\beta,\alpha))^{\vee}} \frac{q^{\overline{2\alpha}} - q^{-\overline{2\alpha}}}{q^{\bar{2}} - q^{-\bar{2}}} .$$

where we denote $a\bar{b}$ *by* $(a/b)^{\vee}$ *and* $\bar{\alpha}$ *is the inverse of* α *in* $\mathbb{Z}/2r\mathbb{Z}$.

Remark 5.3. As far as seeing the above values of lens spaces, our invariant $\tau(M)$ describes the asymptotic behavior of quantum $SO(3)$ invariant $\tau_r'(M)$. By Lemma 5.4 below, $\tau_r'(\text{Ł}(\alpha,\beta))$ has a rational expression in $q^{\overline{2\alpha}}$, recall that q is $\exp(2\pi\sqrt{-1}/r)$ and $\overline{2\alpha}$ is the inverse of 2α in $\mathbb{Z}/r\mathbb{Z}$. The series $q^{\overline{2\alpha}}$ $(r = 3,5,7,11,\dots)$ splits into subsequences, each of which is a series converges to $\zeta_k = \exp(k\pi\sqrt{-1}/\alpha)$, because $(q^{\overline{2\alpha}})^{2\alpha} = q$ converges to 1. According to the splitting, the series $\tau_r'(L(\alpha,\beta))$ $(r = 3,5,7,11,\dots)$ splits into subsequences, each of which follows the asymptotic behavior of $\pm\tau(L(\alpha,\beta))$ around $t^{1/2\alpha} = \zeta_k$. Namely, we can obtain the asymptotic formula of $\tau_r'(L(\alpha,\beta))$ by expanding $\pm\tau(L(\alpha,\beta))$ at $t^{1/2\alpha} = \zeta_k$.

Hence $\pm\tau(L(\alpha,\beta))$ is a formula which was sought by Freed–Gompf [4] and Jeffrey [6]; they investigated the asymptotic formula of quantum $SU(2)$ invariants of lens spaces in order to compare the formula with the asymptotic formula predicted by perturbation of path integral. Note that we can express quantum $SU(2)$ invariant using quantum $SO(3)$ invariant, see Remark 2.2.

Lemma 5.4. *If* α *is odd, then we have*

$$3\cdot s(\beta,\alpha) \equiv 0 \mod \frac{1}{2\alpha} .$$

Proof.

$$\begin{aligned} 3\cdot s(\beta,\alpha) &\equiv 3\sum_{k=1}^{\alpha-1}\frac{k}{\alpha}\left(\frac{k\beta}{\alpha} - \frac{1}{2}\right) \\ &= \frac{3\beta}{\alpha^2}\cdot\frac{1}{6}\alpha(\alpha-1)(2\alpha-1) - \frac{3}{2\alpha}\cdot\frac{1}{2}\alpha(\alpha-1) \\ &= \frac{1}{4\alpha}(\alpha-1)(2\beta(2\alpha-1)-3\alpha) \end{aligned}$$

Since $\alpha - 1$ is even, we have the required formula. □

Remark 5.5. Proposition 5.2 was generalized by Takata [18] to quantum $PSU(N)$ invariant. Her formula is as follows.

$$\tau_r^{PSU(N)}(L(\alpha,\beta)) = \left(\frac{\alpha}{r}\right)^{N-1} q^{-(N(N^2-1)\cdot s(\beta,\alpha)/2)^{\vee}} \frac{[\bar{\alpha}]^{N-1}[2\bar{\alpha}]^{N-2}\cdots[(N-1)\bar{\alpha}]}{[1]^{N-1}[2]^{N-2}\cdots[N-1]}$$

By this formula we could expect that the same argument might go for quantum $PSU(N)$ invariant. If we could define a polynomial invariant "$\tau^{PSU(N)}(M)$", it must satisfy

$$\tau^{PSU(N)}(L(\alpha,\beta)) = t^{-N(N^2-1)\,\cdot\, s(\beta,\alpha)/2}\frac{[1/\alpha]_t^{N-1}[2/\alpha]_t^{N-2}\cdots[(N-1)/\alpha]_t}{[1]_t^{N-1}[2]_t^{N-2}\cdots[N-1]_t}$$

$$= \alpha^{-N(N-1)/2}\left(1 - \frac{N(N^2-1)}{2}s(\beta,\alpha)(t-1)+\cdots\right)$$

where $[k]_t = (t^{k/2} - t^{-k/2})/(t^{1/2} - t^{-1/2})$. This formula suggests that "$SU(N)$ Casson invariant $\lambda^{SU(N)}(M)$" might satisfy

$$\lambda^{SU(N)}(L(\alpha,\beta)) = -\frac{N(N^2-1)}{12}s(\beta,\alpha)$$

$$= \binom{N+1}{3}\lambda(L(\alpha,\beta))$$

if we could define it.

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